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ON A NOVEL APPROACH TO THE PROBLEM OF THE CONCENTRATION OF ELASTIC STRESSES NEAR A CRACK*

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The three-dimensional problem of the theory of elasticity dealing with the concentration of stresses near plane cracks is used to describe a novel approach to solving the problem. The proposed approach is based, unlike the traditional approach, on obtaining an expression for the stress field as simple as that for the displacements, but using new harmonic functions connected, in a prescribed manner, with the normally used harmonic functions of the displacement field. In the case of a plane plane crack the new representation for the stress field leads to three-dimensional integral equations of a single type and solvable separately, replacing the system of three integrodifferential equations of the traditional approach. Moreover, the simple integral equations obtained must be solved in a wider than usual class of functions, namely on the class of functions containing non-integrable singularities. All this leads to a significant simplification of the formula for the stress intensity coefficient.

1. ***Modification of the Trefftz solution of the Lamé equations.*** If we assume that $v = 2Gu$ where $u(x, y, z)$ is the displacement vector with coordinates $u_j(x, y, z)$ $(j = 1, 2, 3)$ and G is the shear modulus, then we can represent the solution of the homogeneous Lamé equations in the form /1/

$$v = \psi + x \operatorname{grad} \psi_0 \tag{1.1}$$

where ψ is a harmonic vector with coordinates $\psi_j\,(j = 1, 2, 3)$ and ψ_0 is a harmonic function. Here the harmonic functions $\psi_j\,(j = 0, 1, 2, 3)$ are connected by the equation

$$\varkappa\psi_0' = -\operatorname{div}\psi,\ \varkappa = 3 - 4\mu \tag{1.2}$$

where μ is Poisson's ratio and a prime denotes a derivative with respect to x.

In accordance with formula (1.1) and using the Hooke's law, we establish the following formulas for the stresses:

$$\sigma_x = (1-2\mu)\psi_0' + \psi_1' + x\psi_0'',\quad 2\tau_{xy} = \psi_1^{\cdot} + \psi_0^{\cdot} + \psi_2' + 2x\psi_0'^{\cdot}$$
$$2\tau_{xz} = \psi_1{}^{,} + \psi_0{}^{,} + \psi_3' + 2x\psi_0'{}^{,} \tag{1.3}$$

A dot denotes differentiation with respect to y, and a comma denotes differentiation with respect to z.

Let us introduce a new harmonic vector ψ^* with coordinates $\psi_j^*, (j = 1, 2, 3)$, defined by the formulas

$$\psi_1^* = (1-2\mu)\psi_0' + \psi_1',\quad 2\psi_2^* = \psi_1^{\cdot} + \psi_0^{\cdot} + \psi_2'$$
$$2\psi_3^* = \psi_1{}^{,} + \psi_0{}^{,} + \psi_3' \tag{1.4}$$

The harmonicity of the functions $\psi_j^*\ (j = 1, 2, 3)$ follows from the harmonicity of ψ_j $(j = 0, 1, 2, 3)$.

Using formulas (1.4) and taking Eqs.(1.2) into account, we establish that

$$\operatorname{div}\psi^* = -\psi_0'' \tag{1.5}$$

If the vector ψ^* has been found, then relation (1.5) can be used as the equation for determining ψ_0. Using the harmonic functions $\psi_j\ (j = 1, 2, 3)$, we can express the stresses (1.3) by the formulas

$$\sigma_x = \psi_1^* + x\psi_0'',\quad \tau_{xy} = \psi_2^* + x\psi_0'^{\cdot},\quad \tau_{xz} = \psi_3^* + x\psi_0'{}^{,} \tag{1.6}$$

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which are as simple as those for the displacements (1.1).

2. ***Reducing the problem of a crack in the plane*** $x = 0$ ***to two-dimensional integral equations.*** We shall assume that the crack occupies the region S, which may be multiply connected, in the plane $x = 0$, and the elastic unbounded medium with the constants μ and G is arbitrarily loaded. We will assume that the stresses are caused by the loads in question and when there is no crack, have been found and are given by the formulas

$$\sigma_x = q_1(x, y, z), \quad \tau_{xy} = q_2(x, y, z), \quad \tau_{xz} = q_3(x, y, z) \tag{2.1}$$

We shall seek the stress distribution in an unbounded elastic medium containing the crack S, in the form of the sum of two terms

$$\sigma_x = q_1 + \sigma_x^*, \quad \tau_{xy} = q_2 + \tau_{xy}^*, \quad \tau_{xz} = q_3 + \tau_{xz}^* \tag{2.2}$$

Here an asterisk denotes the discontinuous displacements and stresses /2/ caused by the presence of the crack S. These must remove the stresses from the edges of the crack $x = \pm 0$ appearing there in accordance with formula (2.1), i.e. they must satisfy, at these edges, the following boundary conditions:

$$x = \pm 0, \quad \sigma_x^* = -q_1(0, y, z), \quad \tau_{xy}^* = -q_2(0, y, z), \quad \tau_{xz}^* = -q_3(0, y, z), \quad y, z \in S \tag{2.3}$$

We shall construct the discontinuous stress and displacement fields using formulas (1.6). In order to ensure the discontinuity, we must construct a discontinuous harmonic function $\psi(x, y, z)$ with the jumps

$$\begin{aligned} [\psi(0, y, z)] &= \psi(-0, y, z) - \psi(+0, y, z) \\ [\psi'(0, y, z)] &= \psi'(-0, y, z) - \psi'(+0, y, z) \end{aligned} \tag{2.4}$$

on S. We shall use the well-known scheme /2/, first introducing a double Fourier transformation in the variables y and z:

$$\psi_{\beta\lambda}(x) = \iint_{-\infty}^{\infty} \psi(x, y, z)\, e^{i\beta y + i\lambda z}\, dy\, dz$$

Then, the double Fourier transform of the discontinuous harmonic function sought will be given by the formula

$$\begin{aligned} \psi_{\beta\lambda}(x) &= [\psi_{\beta\lambda}'(0)]\, \Phi_{\beta\lambda}(x) + [\psi_{\beta\lambda}(0)]\, \Phi_{\beta\lambda}'(x) \\ \Phi_{\beta\lambda}(x) &= (2\sqrt{\beta^2 + \lambda^2})^{-1} e^{-|x|\sqrt{\beta^2+\lambda^2}} \end{aligned} \tag{2.5}$$

Here $[\psi_{\beta\lambda}(0)]$, $[\psi_{\beta\lambda}'(0)]$ are the double Fourier transforms of the jumps (2.4). Inverting the resulting transform (2.5), we obtain the required discontinuous harmonic function with jumps (2.4),

$$\begin{aligned} 4\pi\psi(x, y, z) &= \frac{1}{\pi} \iint_{-\infty}^{\infty} \psi_{\beta\lambda}(x)\, e^{-i\beta y - i\lambda z}\, d\beta\, d\lambda = \\ &\iint_S \frac{[\psi'(0, \eta, \zeta)]\, d\eta\, d\zeta}{R_x(y, z, \eta, \zeta)} + \frac{\partial}{\partial x} \iint_S \frac{[\psi(0, \eta, \zeta)]\, d\eta\, d\zeta}{R_x(y, z, \eta, \zeta)} \\ R_x(y, z, \eta, \zeta) &= \sqrt{x^2 + (y - \eta)^2 + (z - \zeta)^2} \end{aligned} \tag{2.6}$$

Let us take, as the functions $\psi_j^*(x, y, z)$ appearing in (2.2), the discontinuous harmonic functions determined by the formula (2.6), and let us realize the conditions (2.3). The latter conditions imply that the jumps in the values of the stresses during the passage across the surface S are zero, therefore according to (1.6) we have

$$[\psi_j^*(0, y, z)] = 0, \quad j = 1, 2, 3 \tag{2.7}$$

Therefore by virtue of (2.6) the harmonic functions ψ_j^* will be given by the formulas

$$\psi_j^*(x, y, z) = \frac{1}{4\pi} \iint_S \frac{[\psi_j^{*\prime}(0, \eta, \zeta)]}{R_x(y, z, \eta, \zeta)}\, d\eta\, d\zeta, \quad j = 1, 2, 3 \tag{2.8}$$

and from this it follows that the boundary conditions lead, according to (1.6), to the following integral equations for determining the unknown jumps $[\psi_j'(0, \eta, \zeta)]$:

$$\frac{1}{4\pi}\iint_S \frac{[\psi_j^{*\prime}(0, \eta, \zeta)]\, d\eta d\zeta}{R_0(y, z, \eta, \zeta)} = -q_j(0, y, z), \quad j = 1, 2, 3, \quad y, z \in S \tag{2.9}$$

Having found the jumps, we can determine the functions $\psi_j(x, y, z)$ $(j = 1, 2, 3)$ using the formulas (2.8).

In order to determine the stresses at any point of the unbounded elastic medium, we must also find, according to (1.6), the harmonic function $\psi_0(x, y, z)$. We shall use the following scheme to achieve this. We can obtain the double Fourier transform $\psi_0(x, y, z)$ using formula (2.5), i.e.

$$\psi_{0\beta\lambda}(x) = [\psi_{0\beta\lambda}'(0)]\, \Phi_{\beta\lambda}(x) + [\psi_{0\beta\lambda}(0)]\, \Phi_{\beta\lambda}'(x) \tag{2.10}$$

but in this case we must find the jumps $[\psi_{0\beta\lambda}'(0)]$, $[\psi_{0\beta\lambda}(0)]$, after connecting them with the jumps $[\psi_{j\beta\lambda}'(0)]$, $[\psi_{j\beta\lambda}(0)]$.

From (2.7) we have

$$[\psi_{j\beta\lambda}^*(0)] = 0, \; j = 1, 2, 3 \tag{2.11}$$

Taking into account the harmonicity of the function ψ_0, we can write formula (1.5) in the form

$$\psi_1^{*\prime} + \psi_2^{*\bullet} + \psi_3^{*\prime} = \psi_0^{\bullet\bullet} + \psi_0''$$

or, after applying a double Fourier transformation in y and z,

$$\psi_{1\beta\lambda}^{*\prime} - i\beta\psi_{2\beta\lambda}^* - i\lambda\psi_{3\beta\lambda}^* = -(\lambda^2 + \beta^2)\, \psi_{0\beta\lambda} \tag{2.12}$$

Changing now to the jumps, we establish the relation

$$[\psi_{0\beta\lambda}(0)] = (\lambda^2 + \beta^2)^{-1} \{i\beta\, [\psi_{2\beta\lambda}^*(0)] + i\lambda\, [\psi_{3\beta\lambda}^*(0)] - [\psi_{1\beta\lambda}^{*\prime}(0)]\}$$

Let us differentiate relation (2.12) in x, and take into account the fact that the harmonicity implies $\psi_{1\beta\lambda}^{*\prime\prime} = (\lambda^2 + \beta^2)\, \psi_{1\beta\lambda}^*$. Then

$$[\psi_{0\beta\lambda}'(0)] = (\lambda^2 + \beta^2)^{-1} \{i\beta\, [\psi_{2\beta\lambda}^{*\prime}(0) + i\lambda\, [\psi_{3\beta\lambda}^{*\prime}(0)] - [\psi_{1\beta\lambda}^*(0)]\}$$

If we now take into account formula (2.11), we obtain the following expressions for the jumps:

$$[\psi_{0\beta\lambda}(0)] = -\frac{[\psi_{1\beta\lambda}^{*\prime}(0)]}{\beta^2 + \lambda^2}, \quad [\psi_{0\beta\lambda}'(0)] = \frac{i\beta\, [\psi_{2\beta\lambda}^{*\prime}(0)] + i\lambda\, [\psi_{3\beta\lambda}^{*\prime}(0)]}{\beta^2 + \lambda^2} \tag{2.13}$$

Let us substitute these expressions into formula (2.10) for the transform $\psi_0(x, y, z)$. After inverting the transform with the help of a formula similar to (2.6), we arrive at the relation

$$\begin{aligned} -4\pi\psi_0(x, y, z) = {} & \frac{\partial}{\partial x}\iint_S [\psi_1^{*\prime}(0, \eta, \zeta)]\, L(x, y - \eta, z - \zeta)\, d\eta d\zeta + \\ & \frac{\partial}{\partial y}\iint_S [\psi_2^{*\prime}(0, \eta, \zeta)]\, L(x, y - \eta, z - \zeta)\, d\eta d\zeta + \\ & \frac{\partial}{\partial z}\iint_S [\psi_3^{*\prime}(0, \eta, \zeta)]\, L(x, y - \eta, z - \zeta)\, d\eta d\zeta \end{aligned} \tag{2.14}$$

$$L(x, Y, Z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\exp[-i\beta Y - i\lambda Z - |x|\sqrt{\beta^2 + \lambda^2}]}{(\beta^2 + \lambda^2)^{3/2}}\, d\beta d\lambda.$$

Thus if the integral Eqs.(2.9) have been solved, the stress field in the elastic medium will be given by the formulas (1.6), (2.8) and (2.14).

3. ***On the insufficiency of the solution of integral equation obtained in the class of integrable functions.*** Let us construct a solution of the integral Eq.(2.9) in the class of integrable functions for the case when S is a circle of radius a. We shall show that it is insufficient for the complete solution of the problem. Let us pass to the system of polar coordinates

$$y = r\sin\varphi, \; z = r\cos\varphi, \; \eta = \rho\sin\psi, \; \zeta = \rho\cos\psi \tag{3.1}$$

$$[\psi_j^{*\prime}(0, \rho\sin\psi, \rho\cos\psi)] = \chi_j(\rho, \psi), \; q_j(0, r\sin\varphi, r\cos\varphi) = q_j(r, \varphi)$$

Then in place of (2.9) we shall have

$$\frac{-1}{4\pi}\int_0^a\int_{-\pi}^{\pi}\frac{\chi_j(\rho,\psi)\,\rho d\rho d\psi}{\sqrt{r^2+\rho^2-2r\rho\cos(\varphi-\psi)}}=g_j(r,\varphi),\quad 0\leqslant r\leqslant a,\quad |\varphi|\leqslant\pi \tag{3.2}$$

The above integral equation has several forms of solution in the class of integrable functions (/3, 4/ etc.). The solutions, however, appear to be unsuitable for achieving our purpose. We shall show for example, why the solution of Eq.(3.2) given in /2/ is not suitable. The solution was obtained as follows. The formula 6.511(1) in /5/ was used to show that the kernel of integral Eq.(3.2) can be expressed in terms of the Bessel function of zero order, and the addition formula 8.531(1) in /5/ in which $2\cos n\varphi$ is replaced by $e^{ik\varphi}+e^{-ik\varphi}$, is used as this function. Subsequent application of the finite Fourier transformation given by the formulas

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_j(r,\varphi)\,e^{-in\varphi}d\varphi=\chi_{jn}(r),\quad n=0,\pm1,\pm2,\ldots \tag{3.3}$$

reduces the integral Eq.(3.2) to a sequence of one-dimensional integral equations for the Fourier transforms (after the substitution $r=ax$, $\rho=a\xi$):

$$\int_0^1 W_n(x,\xi)\,\chi_{jn}(a\xi)\,\xi d\xi=-\frac{4\pi}{a}g_{jn}(ax),\quad 0\leqslant x\leqslant 1 \tag{3.4}$$

$$n=0,\ \pm1,\ \pm2,\ \ldots$$

The kernels of these equations are discontinuous Weber-Sonin integrals

$$W_n(x,\xi)=\int_0^\infty J_n(xt)\,J_n(\xi t)\,dt \tag{3.5}$$

for which we have the spectral relation *A* (5.2) of /2/. Using this relation and realizing the method of orthogonal polynomials, we obtain the solution of integral equations in the form /2/

$$\chi_{jn}^0(a\xi)=\sum_{k=0}^\infty\chi_{jnk}\frac{\xi^n}{\sqrt{1-\xi^2}}P_k^{n,-1/2}(1-2\xi^2),\quad j=1,2,3 \tag{3.6}$$

$$\chi_{jnk}=-\frac{16\pi k!^2\,(n+2k+1/2)}{a\Gamma^2(k+1/2)}\int_0^1\frac{g_{jn}(ax)\,x^{n+1}P_k^{n,-1/2}(1-2x^2)\,dx}{\sqrt{1-x^2}} \tag{3.7}$$

$$j=1,\ 2,\ 3;\ n=0,\ 1,\ \ldots;\ k=0,\ 1,\ \ldots$$

Here we have taken into account the fact that, irrespective of the loading of the elastic medium, we can establish a relation connecting g_{jn} and $g_{j,-n}$ (in particular, for an even function $g_j(r,\omega)$ we have, in the variables φ, $g_{j,n}=g_{j,-n}$), therefore we shall have to solve Eqs.(3.4) only for $n\geqslant 0$.

Using the positive definiteness of the kernel we can show that solution (3.6), (3.7) of the integral Eq.(3.4) will be unique in the class of integrable functions. However, this solution (indicated by the superscript zero) does not lead to the solution of our problem. In order to confirm this we shall calculate the normal stress intensity coefficient NSIC $\sigma_x(0,r,\varphi)$. To do this, we shall need the values of these stresses at the continuation of the crack, i.e. when $r>a$. In accordance with formulas (1.6), (2.8) and (3.1) we have, for $-\sigma_x(0,r,\varphi)$, an expression corresponding to the left-hand side of relation (3.2), for $j=1$.

Let us calculate the Fourier transform for the stress $\sigma_x(0,r,\varphi)$, having previously carried out the same operation as when deriving Eq.(3.4). We obtain

$$\sigma_{xn}(ax)=\frac{a}{4\pi}\int_0^1\chi_{1n}(a\xi)\,\xi W_n(x,\xi)\,d\xi,\quad x>1 \tag{3.8}$$

The required NSIC is given by the formula

$$N_1(\varphi)=\lim_{x\to1+0}\sqrt{a(x-1)}\,\sigma_x(0,\ ax,\ \varphi) \tag{3.9}$$

According to the inversion formula the following relation holds for the finite Fourier transformation:

$$\sigma_x(0, ax, \varphi) = \sum_{n=-\infty}^{\infty} \sigma_{xn}(ax) e^{in\varphi}, \quad x \geqslant 1$$

therefore we have

$$N_1(\varphi) = \sum_{n=-\infty}^{\infty} e^{in\varphi} N_1^{(n)}, \quad N_1^{(n)} = \lim_{x\to 1+0} \sqrt{a(x-1)}\,\sigma_{xn}(ax) \tag{3.10}$$

In order to find the limit appearing here, we must use expression (3.9) for $\sigma_{xn}(ax)$ into which the solution (3.6) for $j = 1$ has been previously substituted. Using formula (1.7) of /6/ we have $(x > 1)$ as a result;

$$4\pi\sigma_{xn}(ax) = -ax^{-n-1}S_{1n}(x), \quad S_{1n}(x) = \tag{3.11}$$

$$\frac{-a}{4\pi x^{n+1}} \sum_{k=0}^{\infty} \chi_{1nk} \frac{\Gamma(n+k+{}^1/_2)\,\Gamma(k+{}^1/_2)\,F(n+k+{}^1/_2, k+{}^1/_2; n+2k+{}^3/_2; x^{-2})}{2k!\,\Gamma(n+2k+{}^3/_2)\,\Gamma(-k+{}^1/_2)\,x^{2k}}$$

Continuing analytically the Gauss function appearing in (3.11) to the circumference $x = 1 + 0$, we obtain, using formula 9.131(2) from /5/,

$$\sigma_{xn}(ax) = \frac{a}{8\sqrt{\pi}} \sum_{k=0}^{\infty} \chi_{1kn} \frac{\Gamma(n+k+{}^1/_2)\,\Gamma(k+{}^1/_2)}{\Gamma(-k+{}^1/_2)\,k!\,x^{2k+n+1}} \times$$
$$\{[k!\,(n+k)!]^{-1} F(n+k+{}^1/_2, k+{}^1/_2; {}^1/_2; (x^2-1)x^{-1}) - \sqrt{x^2-1}\,x^{-1}\}\times$$
$$2[\Gamma(n+k+{}^1/_2)\,\Gamma(k+{}^1/_2)]^{-1} F(k+1, n+k+1, n+2k+{}^3/_2; (x^2-1)x^{-1}$$

The above representation shows that the function $\sigma_{xn}(ax)$ is bounded when $x \to 1+0$, and therefore we have $N_1^{(n)} = 0$. Consequently the NSIC (3.9) is equal to zero under any loading of the elastic medium, which is absurd. We arrive at the same result when instead of the solution of integral Eq.(3.2) given in /2/ we use any other solution in the class of integrable functions.

4. The construction of solutions of integral Eq.(3.4) in the class of non-integrable functions. In order to obtain a correct solution of the problem, we must expand the class of functions from which the solutions of integral Eq.(3.2) or Eq.(3.4) must be taken. Let us first consider such an integral:

$$I_n(x, \lambda) = \int_0^1 \frac{\xi^{n+1} W_n(x, \xi)\,d\xi}{(1-\xi^2)^\lambda}, \quad 0 \leqslant x < \infty, \quad n = 0, 1, \ldots \tag{4.1}$$

where λ is a complex number. Substituting into (4.1) the representation (3.5) for the Weber-Sonin integral and changing the order of integration, we obtain

$$I_n(x, \lambda) = \int_0^\infty J_n(tx)\,dt \int_0^1 \frac{\xi^{1+n}}{(1-\xi^2)^\lambda} J_n(t\xi)\,d\xi = \tag{4.2}$$
$$\frac{\Gamma(1-\lambda)}{2^\lambda} \int_0^\infty \frac{J_n(tx)\,J_{n+1-\lambda}(t)}{t^{1-\lambda}}\,dt, \quad 0 \leqslant x < \infty$$

i.e. (4.1) is a Weber-Sonin integral with a discontinuity at the point $x = 1$. We find its value for $x < 1$ and $x > 1$ using formulas 6.574(1) and 6.574(3) from /5/.

Let us continue the result obtained analytically to the value $\lambda = {}^3/_2$. This yields the following values of the integrals, which should be regarded in the generalized sense /7/:

$$\int_0^1 \frac{\xi^{n+1} W_n(x, \xi)}{(1-\xi^2)^{3/2}}\,d\xi = \begin{cases} 0, & 0 \leqslant x < 1 \\ x^{-n}(x^2-1)^{-1/2}, & x > 1 \end{cases} \tag{4.3}$$

The upper relation of (4.3) enables us to construct the solution of integral Eq.(3.4) in the class of non-integrable functions in the form

$$\chi_{jn}(a\xi) = \chi_{jn}^0(a\xi) - C_{jn}\xi^n(1-\xi^2)^{-1/2}$$

where C_{jn} is an arbitrary constant. Substituting the solution obtained into (3.8) and taking into account the upper relation of (4.3) and (3.11), we obtain

$$\sigma_{xn}(ax) = \frac{a}{4\pi x^{n+1}}\left[C_{1n}\frac{x}{\sqrt{x^2-1}} + S_{1n}(x)\right], \quad x > 1 \tag{4.4}$$

We shall use the following arguments to fix the arbitrary constants C_{1n}. Formula (4.4) determines the Fourier transform of the normal stresses at the boundary of the half-space which would form if the unbounded elastic medium were bisected by the plane $x = 0$. In the case of $n = 0$ (the axisymmetric case) $\sigma_{x0}(r)$ will represent the normal stress along the above cut, and it will have to equilibrate the given load applied to the half-space shown, i.e. the resultant of these stresses or the integral

$$\int_1^\infty \sigma_{x0}(r)\, r dr$$

must be finite, and for this we need

$$\sigma_{xn} = O(x^{-n-2-\varepsilon}), \quad x \to \infty, \quad \varepsilon > 0 \tag{4.5}$$

In order to realize this condition, we shall replace the Gauss function in the series appearing in (4.4) by its power representation, and transform the resulting double series to the form

$$S_{1n}(x) = \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)}{x^{2m}} \times \sum_{k=0}^{\infty} \frac{\chi_{1nk}\Gamma(m+n+1/2)}{2k!\,(m-k)!\,\Gamma(-k+1/2)\,\Gamma(m+n+k+3/2)} \tag{4.6}$$

We shall also assume that

$$\frac{x}{\sqrt{x^2-1}} = \sum_{m=0}^{\infty} \frac{(1/2)\, m}{m!} \frac{1}{x^{2m}} \tag{4.7}$$

Taking into account relations (4.6) and (4.7), we will establish that condition (4.5) will hold provided that we assume that

$$C_{1n} = \chi_{1n0}(2n+1)^{-1} \tag{4.8}$$

Let us calculate the NSIC (3.9). In order to use formula (3.10) we must find $N_1^{(n)}$, and we do this by substituting the expression (4.4) into the second relation of (3.10). After carrying out the corresponding passage to the limit, we obtain

$$N_1^{(n)} = -(2\pi)^{-1} C_{1n} (a/2)^{1/2}$$

or, taking into account formulas (4.8) and (3.7),

$$N_1^{(n)} = \frac{\sqrt{2a}}{\pi} \int_0^1 \frac{g_{1n}(ax)\, x^{n+1}}{\sqrt{1-x^2}} dx \tag{4.9}$$

Formulas (3.10) and (4.9) then give the NSIC required.

It is best to compare this result with the analogous result obtained by traditional methods. For example, the NSIC $N_1(\varphi)$ is given, according to /2/, (p.261) after correcting the error, i.e. replacing $e^{-in\varphi}$ by $e^{in\varphi}$, by the same formula (3.10), and unlike (4.9) it has the much more cumbersome form

$$N_1^{(n)} = \sqrt{\frac{2a}{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n+{}^3/_2)\, m!}{\Gamma(m+{}^3/_2)} J^g_{mn} \tag{4.10}$$

$$J^g_{mn} = \int_0^1 g_{1n}(ax)\, P_m^{n,\,{}^1/_2}(1-2x^2)\, x^{n+1}\sqrt{1-x^2}\, dx \tag{4.11}$$

Here it was assumed that $q_n^{(3)}(ax) = -g_{1n}(ax)$.

We shall now show how to reduce this cumbersome formula to the form (4.9). Let us expand the given function g_{1n} in a series in Jacobi polynomials:

$$g_{1n}(ax) = \sum_{k=0}^{\infty} g_{1nk} x^n P_k^{n,\,-{}^1/_2}(1-2x^2) \tag{4.12}$$

Using the orthogonality of the Jacobi polynomials, we obtain

$$g_{1nk} = Q_{nk} g^*_{1nk}, \quad \frac{Q_{nk}}{2} = \frac{k!\,\Gamma(n+k+{}^1/_2)\,\Gamma(n+2k+{}^1/_2)}{\Gamma(n+k+1)\,\Gamma(k+{}^1/_2)} \tag{4.13}$$
$$g^*_{1nk} = \int_0^1 \frac{x^{n+1} g_{1n}(ax)\, P_k^{n,\,-{}^1/_2}(1-2x^2)}{\sqrt{1-x^2}}\, dx$$

Now substituting (4.12) into (4.11), we obtain

$$J^g_{mn} = \sum_{k=0}^{\infty} Q_{nk} g^*_{1nk} I_{mk} \tag{4.14}$$
$$I_{mk} = \int_0^1 P_m^{n,\,{}^1/_2}(1-2x^2)\, P_k^{n,\,-{}^1/_2}(1-2x^2)\, x^{2n+1}\sqrt{1-x^2}\, dx$$

Carrying out the substitution $1-2x^2 = t$ in the last integral, using formula 7.391(9) from /5/ and substituting the result into (4.14), we reduce series (4.14) to a finite sum. As a result formula (4.10) will take the form $N_1^{(n)} = \pi^{-1}\sqrt{2a}\, g^*_{1n0}$. Taking into account the second relation of (4.13), we achieve complete agreement of the results.

5. ***The stress intensity coefficient in the Kelvin problem when there is a disc-shaped crack.*** We shall discuss the results obtained in greater detail, applying them to the case of the loading of an unbounded medium with a concentrated force P (the Kelvin problem), parallel to the x axis and applied to the point with coordinates $x = \xi,\ y = 0,\ z = h$. The circular crack of radius a is situated, as before, in the plane $x = 0$. The distribution of the stresses σ_x at $x = 0$ is given in the Kelvin problem by the formula /8/

$$\sigma_x(0, y, z) = \frac{P\xi}{8\pi(1-\mu)[\xi^2 + y^2 + (z-h)^2]^{3/2}} \times$$
$$\left[1 - 2\mu + \frac{3\xi^2}{\xi^2 + y^2 + (z-h)^2}\right] = q_1(0, y, z)$$

Changing the variables (3.1) and passing to dimensionless coordinates $r = ax\ \xi = ab,\ h = ac$, we obtain

$$g_{1n}(ax) = -\frac{P}{4\pi a^2}\left[I_n{}'(b, c; x) - \frac{b}{2(1-\mu)} I_n{}''(b, c; x)\right] \tag{5.1}$$
$$I_n(b, c; x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\varphi}\, d\varphi}{\sqrt{b^2 + c^2 + x^2 - 2cx\cos\varphi}} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos n\varphi\, d\varphi}{\sqrt{b^2 + c^2 + x^2 - 2cx\cos\varphi}}$$

(the prime denotes a derivative with respect to b). The last integral can be calculated as follows. We make the substitution $\cos\varphi = t$ and use the representation $T_n(t) = \cos n \arccos t$ for the Chebyshev polynomial. As a result we obtain

$$\pi I_n(b, c; x) = \frac{1}{\sqrt{b^2 + (c+x)^2}} \int_{-1}^{1} \frac{T_n(t)\, dt}{\sqrt{1-t^2}\sqrt{1 - y(1+t)}}, \quad y = \frac{2cx}{b^2 + (c+x)^2}$$

Expanding the radical in the integrand in a Maclaurin series in y and evaluating the remaining integrals with the help of formula 7.391(3) from /5/, we obtain

$$\sqrt{\pi}\, I_n(b, c; x) = \frac{\Gamma(n + 1/2)(cx)^n}{[b^2 + (c+x)^2]^{n+1/2}} F\left(n + \frac{1}{2}, n + \frac{1}{2}; 2n+1; \frac{4cx}{b^2 + (c+x)^2}\right) \tag{5.2}$$

When determining the NSIC we should remember that in the present case we have $g_{1,n} = g_{1,-n}$ by virtue of (5.1), and therefore $N_1^{(n)} = N_1^{(-n)}$. Formula (3.10) will then become

$$N_1(\varphi) = N_1^{(0)} + 2\sum_{n=1}^{\infty} N_1^{(n)} \cos n\varphi \tag{5.3}$$

The coefficients $N_1^{(n)}$ can be found, according to (4.9) and (5.1), from the formula

$$N_1^{(n)} = -\frac{P\pi^{-2}}{(2a)^{3/2}}\left[J_n'(b, c) - \frac{b}{2(1-\mu)} J_n''(b, c)\right] \tag{5.4}$$

$$J_n(b, c) = \int_0^1 \frac{I_n(b, c; x)\, x^{n+1}}{\sqrt{1-x^2}}\, dx \tag{5.5}$$

Next we expand the integral (5.6) in a power series in c, using the representation

$$\frac{b^{2\lambda}}{[b^2 + (c+x)^2]^{\lambda}} = \sum_{m=0}^{\infty} c_m \left(i\frac{c+x}{b}\right)^n, \quad c_m = \sum_{j=0}^{m} \frac{(\lambda)_j (\lambda)_{m-j} (-1)^j}{j!\,(m-j)!} =$$
$$(\lambda)_m (m!)^{-1} F(-m, \lambda; 1-\lambda-m; -1) =$$
$$(-2)^m (\pi^{3/2} m!)^{-1} \Gamma(1-\lambda)\, \Gamma(\lambda + 1/2m)\, \Gamma(1/2 + 1/2m) \cos 1/2m\pi \sin(\lambda + 1/2m)\pi \tag{5.6}$$

where we have used the well-known formula (/9/, p.112) for Gauss's function with argument equals to -1.

The last equation yields important relations

$$c_{2k} = (\lambda)_k (k!)^{-1}, \quad c_{2k+1} = 0, \quad k = 0, 1, \ldots \tag{5.7}$$

Using in (5.6) the formula for Newton's binormal and taking into account relations (5.7), we obtain

$$\frac{b^{2\lambda}}{[b^2 + (c+x)^2]^{\lambda}} = \sum_{j=0}^{\infty} \frac{c^{2j}}{(2j)!} \sum_{k=j}^{\infty} \frac{(\lambda)_k}{k!} \frac{(-1)^k}{b^{2k}} (-2k)_{2j}\, x^{2(k-j)} -$$
$$\sum_{j=0}^{\infty} \frac{c^{2j+1}}{(2j+1)!} \sum_{k=j+1}^{\infty} \frac{(\lambda)_k}{k!} \frac{(-1)^k}{b^{2k}} (-2k)_{2j+1} x^{2k-2j-1} \tag{5.8}$$

If we now substitute expression (5.2) into (5.5) having previously replaced Gauss's function in it by its representation in the form of a power series and using Eq.(5.8), we obtain the required power representation

$$J_n(b, c) = \frac{\sqrt{\pi}}{4} \sum_{j=0}^{\infty} \frac{(n + 1/2)_j^2 (4c)^j}{(2n+1)_j\, j!} \left[\sum_{k=0}^{\infty} \frac{c^{2k}}{(2k)!} B_{jk}^{n+}(b) + \sum_{k=0}^{\infty} \frac{c^{2k+1}}{(2k+1)!} B_{jk}^{n-}(b)\right] \tag{5.9}$$

$$B_{jk}^{n+} = (-1)^k 2^{2k} b^{-2n-2k-2j-1} (1/2)_k (n + j + 1/2)_k \Gamma(n + 1/2j + 1)\, \Gamma^{-1}(n + 1/2j +$$
$$5/2)\, {}_3F_2(n + k + j + 1/2,\ k + 1/2,\ n + 1/2j + 1;\ 1/2,\ n + 1/2j + 5/2;\ -b^{-2})$$

$$B_{jk}^{n-} = (-1)^k 2^{2k+1} b^{-2n-2k-2j-3} (3/2)_k (n + j + 1/2)_k \Gamma(n + 1/2j + 3/2) \times$$
$$\Gamma^{-1}(n + 1/2j + 7/2)\, {}_3F_2(n + k + j + 3/2,\ k + 3/2,\ n + 1/2j + 3/2;\ 3/2,\ n + 1/2j +$$
$$7/2;\ -b^{-2})$$

Thus the NSIC required is given by (5.3), (5.4) and (5.9), which can be used to find the NSIC for the case of axial symmetry. To do this it is sufficient to put $c = 0$ in formulas (5.4) and (5.9). As a result we find

$$N_1^{(0)} = -1/3\pi^{-2}(1-\mu)^{-1} P (2a)^{-3/2} b^{-2} [(\mu - 2) F(1, 1/2; 5/2; -b^{-2}) +$$
$$1/5(7 - 2\mu) b^{-2} F(2, 3/2; 7/2; -b^{-2}) - 12/35\, b^{-4} F(3, 5/2; 9/2; -b^{-2})] \tag{5.10}$$

It is best to use the above formula when $b \gg 1$, otherwise we have to apply formula 9.131 and 9.132 from /5/. In order to demonstrate their use, we will obtain the NSIC for the case when $b=0$, i.e. for the case when the force P is applied directly to the edge $x=+0$ of the crack. In this case Gauss's function in (5.10) should be transformed using formula 9.132(2) from /5/, and this will enable us to carry out, after some reduction, the passage to the limit as $b \to 0$. As a result we obtain

$$N_1^{(0)} = \pi^{-2} P\,(2a)^{-3/2}\,(8-3\mu)\,(1-\mu)^{-1}.$$

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FRACTURE OF A NARROW BRIDGE BETWEEN CRACKS LYING IN THE SAME PLANE*

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The stress-deformation state of an isotropic elastic space weakened by a family of cracks of normal separation is investigated. In some regions the crack edges come closer to each other, and form narrow bridges (ligaments). An asymptotic form of the solution of the problem is constructed under the assumption that the bridge either contracts to a contour, or becomes an open arc. Special features of the stresses at the tip of the bridge are studied for various forms of the tip. Asymptotic formulas obtained are used to produce variational formulations of the problems, and the lack of uniqueness of these solutions is interpreted as the instability of the process of disruption of narrow bridges. Examples are considered.

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